Indian Statistical Institute, Bangalore Centre
B.Math (Hons.) I Year, First Semester
Mid-Sem Examination
Analysis I

Time: 150 Minutes September 29, 2010 Instructor: C.R.E.Raja

Answer any five and each question is worth 8 marks. Total Mark is 40.

1. Let (x_n) be a sequence of real numbers.

   (a) Show that (x_n) converges implies (x_n) is bounded.

   (b) Let $x \in I\!R$. Show that $x_n \to x$ iff $\liminf x_n = x = \limsup x_n$.

2. (a) Say true or false: every bounded sequence (x_n) has a subsequence (x_{k_n}) such that $x_{k_n} \to \limsup x_n \in I\!R$.

   (b) Prove that every bounded sequence has a convergent subsequence.

   (c) Suppose (a_n) and (b_n) are bounded sequences in $I\!R$. Then prove that (a_n+b_n) is a bounded sequence and $\limsup(a_n+b_n) \leq \limsup a_n + \limsup b_n$.

3. (a) If $|x| < 1$ and $q \in I\!R$, show that $\sum n^q x^n$ converges.

   (b) Prove that $n^{\frac{1}{n}} \to 1$ and use it to show that $\sum a_n$ converges where $a_n = (n^{\frac{1}{n}} - 1)^n$.

4. Let $\sum a_n$ converge absolutely. Then show that $\sum a_n$ converges and every rearrangement $\sum a_{k_n}$ also converges to $\sum a_n$. Show also that any rearrangement $\sum a_{k_n}$ converges absolutely and $\sum |a_n| = \sum |a_{k_n}|$.

5. (a)Let A be a non-empty subset of the real line $I\!R$ and define $f\colon I\!R \to I\!R$ by $f(x) = \inf_{a\in A} |x-a|$, $x \in I\!R$. Show that f is continuous on $I\!R$.

   (b) Let f be a continuous function on $I\!R$ and $x \in I\!R$. Suppose $f(x) \neq 0$. Show that there are $\delta, \eta > 0$ such that $|f(r)| > \eta$ for all $r \in (x-\delta, x+\delta)$.

   (c) Let f and g be continuous functions on $I\!R$ and $x \in I\!R$. Suppose $f(x) \neq g(x)$. Show that there is a $\delta > 0$, such that $f(r) \neq g(r)$ for all $r \in (x-\delta, x+\delta)$.

6. (a) If $f\colon [0,1] \to [0,1]$ is a continuous function, then prove that there is a $x \in [0,1]$ such that $f(x) = x$.

   (b) If $f\colon [0,1] \to [0,2]$ is a continuous function, then prove that there is a $x \in [0,1]$ such that $f(x) = 2-2x$.

7. Let $f\colon I \to I\!R$ be an uniformly continuous function and (x_n) be a sequence in I.

   (a) If (x_n) is Cauchy, show that $(f(x_n))$ is Cauchy.

(b) If (x_n) and (y_n) are sequences in I such that $x_n \to x$, $y_n \to x$ for some $x \in \mathbb{R}$, then show that $\lim f(x_n) = \lim f(y_n)$.